

# Cohomology and Formal Deformations of Left Alternative Algebras

Mohamed Elhamdadi  
Department of Mathematics  
and Statistics  
University of South Florida  
emohamed@math.usf.edu

Abdenacer Makhoulf  
Laboratoire de Mathématiques,  
Informatique et Applications  
Université de Haute-Alsace  
Abdenacer.Makhoulf@uha.fr

July 9, 2009

## Abstract

The purpose of this paper is to introduce an algebraic cohomology and formal deformation theory of left alternative algebras. Connections to some other algebraic structures are given also.

## 1 Introduction

Deformation theory arose mainly from geometry and physics. In the later field, the non-commutative associative multiplication of operators in quantum mechanics is thought of as a formal associative deformation of the pointwise multiplication of the algebra of symbols of these operators. In the sixties, Murray Gerstenhaber introduced algebraic formal deformations for associative algebras in a series of papers (see [12, 13, 14, 15]). He used formal series and showed that the theory is intimately connected to the cohomology of the algebra. The same approach was extended to several algebraic structures. Other approaches to study deformations exist, see [9, 10, 11, 18, 19, 22], see [23] for a review.

In this paper we introduce a cohomology and a formal deformation theory of left alternative algebras. We also review the connections of alternative algebras to other algebraic structures. In Section 2, we review the basic definitions and properties related to alternative algebras. In Section 3, we discuss in particular all the links between alternative algebras and some other algebraic structures such as Moufang loops, Malcev algebras, Jordan algebras and Yamaguti-Lie algebras called also generalized Lie triple systems. In Section 4, we introduce a cohomology theory of left alternative algebras. We compute the second cohomology group of the 2 by 2 matrix algebra. It is known that, as an associative algebra its second cohomology group is trivial, but we show that this is not the case as left alternative algebra. Finally, in Section 5, we consider the formal deformation theory of left alternative algebras.

## 2 Preliminaries

Throughout this paper  $\mathbb{K}$  is an algebraically closed field of characteristic 0.

## 2.1 Definitions

**Definition 2.1** [28] A left alternative  $\mathbb{K}$ -algebra (resp. right alternative  $\mathbb{K}$ -algebra)  $(\mathcal{A}, \mu)$  is a vector space  $\mathcal{A}$  over  $\mathbb{K}$  and a bilinear multiplication  $\mu$  satisfying the left alternative identity, that is, for any  $x, y \in \mathcal{A}$ ,

$$\mu(x, \mu(x, y)) = \mu(\mu(x, x), y). \quad (1)$$

respectively, right alternative identity, that is

$$\mu(\mu(x, y), y) = \mu(x, \mu(y, y)). \quad (2)$$

An alternative algebra is one which is both left and right alternative algebra.

**Lemma 2.2** Let  $\mathfrak{as}$  denotes the associator, which is a trilinear map defined by  $\mathfrak{as}(x, y, z) = \mu(\mu(x, y), z) - \mu(x, \mu(y, z))$ . An algebra is alternative if and only if the associator  $\mathfrak{as}(x, y, z)$  is an alternating function of its arguments, that is

$$\mathfrak{as}(x, y, z) = -\mathfrak{as}(y, x, z) = -\mathfrak{as}(x, z, y) = -\mathfrak{as}(z, y, x)$$

This lemma implies then that the following identities are satisfied

$$\begin{aligned} \mathfrak{as}(x, x, y) &= 0 \quad (\text{left alternativity}), \\ \mathfrak{as}(y, x, x) &= 0 \quad (\text{right alternativity}) \\ \mathfrak{as}(x, y, x) &= 0 \quad (\text{flexibility}). \end{aligned}$$

By linearization, we have the following characterization of left (resp. right) alternative algebras, which will be used in the sequel.

**Lemma 2.3** A pair  $(\mathcal{A}, \mu)$  is a left alternative  $\mathbb{K}$ -algebra (resp. right alternative  $\mathbb{K}$ -algebra) if and only if the identity

$$\mu(x, \mu(y, z)) - \mu(\mu(x, y), z) + \mu(y, \mu(x, z)) - \mu(\mu(y, x), z) = 0. \quad (3)$$

respectively,

$$\mu(x, \mu(y, z)) - \mu(\mu(x, y), z) + \mu(x, \mu(z, y)) - \mu(\mu(x, z), y) = 0. \quad (4)$$

holds.

**Remark 2.4** When considering multiplication as a linear map  $\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ , the condition (3) (resp. (4)) may be written

$$\mu \circ (\mu \otimes \text{id} - \text{id} \otimes \mu) \circ (\text{id}^{\otimes 3} + \sigma_1) = 0. \quad (5)$$

respectively

$$\mu \circ (\mu \otimes \text{id} - \text{id} \otimes \mu) \circ (\text{id}^{\otimes 3} + \sigma_2) = 0. \quad (6)$$

where  $\text{id}$  stands for the identity map and  $\sigma_1$  and  $\sigma_2$  stands for transpositions generating the permutation group  $\mathcal{S}_3$  which are extended to trilinear maps defined by,  $\sigma_1(x_1 \otimes x_2 \otimes x_3) = x_2 \otimes x_1 \otimes x_3$  and  $\sigma_2(x_1 \otimes x_2 \otimes x_3) = x_1 \otimes x_3 \otimes x_2$  for all  $x_1, x_2, x_3 \in \mathcal{A}$ . In terms of associators, the identities (3) (resp. (4)) are equivalent to

$$\mathfrak{as} + \mathfrak{as} \circ \sigma_1 = 0 \quad (\text{resp. } \mathfrak{as} + \mathfrak{as} \circ \sigma_2 = 0.) \quad (7)$$

**Remark 2.5** The notions of subalgebra, ideal, quotient algebra are defined in the usual way. For general theory about alternative algebras (see [28]). The alternative algebras generalize associative algebras. Recently, in [8], it was shown that their operad is not Koszul. The dual operad of right alternative (resp. left alternative) algebras is defined by associativity and the identity  $\mu(\mu(x, y), z) + \mu(\mu(x, z), y) = 0$ , (resp.  $\mu(\mu(x, y), z) + \mu(\mu(y, x), z) = 0$ ). The dual operad of alternative algebras is defined by the associativity and the identity

$$\mu(\mu(x, y), z) + \mu(\mu(y, x), z) + \mu(\mu(z, x), y) + \mu(\mu(x, z), y) + \mu(\mu(y, z), x) + \mu(\mu(z, y), x) = 0.$$

## 2.2 Structure theorems and Examples

We have these following fundamental properties:

- **Artin's theorem.** In an alternative algebra the subalgebra generated by any two elements is associative. Conversely, any algebra for which this is true is clearly alternative. It follows that expressions involving only two variables can be written without parenthesis unambiguously in an alternative algebra.
- **Generalization of Artin's theorem.** Whenever three elements  $x, y, z$  in an alternative algebra associate (i.e.  $as(x, y, z) = 0$ ), the subalgebra generated by those elements is associative.
- **Corollary of Artin's theorem.** Alternative algebras are power-associative, that is, the subalgebra generated by a single element is associative. The converse need not hold: the sedenions are power-associative but not alternative.

**Example 2.6 (4-dimensional Alternative algebras.)** According to [16], p 144, there are exactly two alternative but not associative algebras of dimension 4 over any field. With respect to a basis  $e_0, e_1, e_2, e_3$ , one algebra is given by the following multiplication (the unspecified products are zeros)

$$e_0^2 = e_0, e_0e_1 = e_1, e_2e_0 = e_2, e_2e_3 = e_1, e_3e_0 = e_3, e_3e_2 = -e_1.$$

The other algebra is given by

$$e_0^2 = e_0, e_0e_2 = e_2, e_0e_3 = e_3, e_1e_0 = e_1, e_2e_3 = e_1, e_3e_2 = -e_1.$$

**Example 2.7 (Octonions)** The octonions were discovered in 1843 by John T. Graves who called them Octaves and independently by Arthur Cayley in 1845. The octonions algebra which is also called Cayley Octaves or Cayley algebra is an 8-dimensional algebra defined with respect to a basis  $u, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ , where  $u$  is the identity for the multiplication, by the following multiplication table. The table describes multiplying the  $i$ th row elements by the  $j$ th column elements.

	$u$	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$u$	$u$	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$e_1$	$e_1$	$-u$	$e_4$	$e_7$	$-e_2$	$e_6$	$-e_5$	$-e_3$
$e_2$	$e_2$	$-e_4$	$-u$	$e_5$	$e_1$	$-e_3$	$e_7$	$-e_6$
$e_3$	$e_3$	$-e_7$	$-e_5$	$-u$	$e_6$	$e_2$	$-e_4$	$e_1$
$e_4$	$e_4$	$e_2$	$-e_1$	$-e_6$	$-u$	$e_7$	$e_3$	$-e_5$
$e_5$	$e_5$	$-e_6$	$e_3$	$-e_2$	$-e_7$	$-u$	$e_1$	$e_4$
$e_6$	$e_6$	$e_5$	$-e_7$	$e_4$	$-e_3$	$-e_1$	$-u$	$e_2$
$e_7$	$e_7$	$e_3$	$e_6$	$-e_1$	$e_5$	$-e_4$	$-e_2$	$-u$

The octonion algebra is a typical example of alternative algebras. As stated early the subalgebra generated by any two elements is associative. In fact, the subalgebra generated by any two elements of the octonions is isomorphic to the algebra of reals  $\mathbb{R}$ , the algebra of complex numbers  $\mathbb{C}$  or the algebra of quaternions  $\mathbb{H}$ , all of which are associative. See [4] for the role of the octonions in algebra, geometry and topology and see also [1] where octonions are viewed as quasialgebra.

### 3 Connections to other algebraic structures

We begin by recalling some basics of *Moufang* loops, Moufang algebras and Malcev algebras.

**Definition 3.1** [30] Let  $(M, *)$  be a set with a binary operation. It is called a *Moufang loop* if it is a quasi-group with an identity  $e$  such that the binary operation satisfies one of the following equivalent identities:

$$x * (y * (x * z)) = ((x * y) * x) * z, \quad (8)$$

$$z * (x * (y * x)) = ((z * x) * y) * x, \quad (9)$$

$$(x * y) * (z * x) = (x * (y * z)) * x. \quad (10)$$

The typical examples include groups and the set of nonzero octonions which gives nonassociative Moufang loop.

As in the case of Lie group, there exists a notion of analytic Moufang loop [29, 30, 26]. An analytic Moufang loop  $M$  is a real analytic manifold with the multiplication and the inverse,  $g \mapsto g^{-1}$ , being analytic mappings. The tangent space  $T_e M$  is equipped with a skew-symmetric bracket  $[\cdot, \cdot] : T_e M \times T_e M \rightarrow T_e M$  satisfying the Malcev's identity that is

$$[J(x, y, z), x] = J(x, y, [x, z]) \quad (11)$$

for any  $x, y, z \in T_e M$  and where  $J$  corresponds to Jacobi's identity i.e.

$$J(x, y, z) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]].$$

**Definition 3.2** [21] A *Malcev*  $\mathbb{K}$ -algebra is a vector space over  $\mathbb{K}$  and a skew-symmetric bracket satisfying the identity (11).

The Malcev algebras are also called Moufang-Lie algebras. We have the following fundamental Kerdman's theorem [21]:

**Theorem 3.3 (Kerdman)** *For any real Malcev algebra there exists an analytic Moufang loop whose tangent algebra is the given Malcev algebra.*

The connection to alternative algebras is given by the following proposition:

**Proposition 3.4** *The alternative algebras are Malcev-admissible algebras, that is the commutators define a Malcev algebra.*

**Remark 3.5** Let  $\mathcal{A}$  be an alternative algebra with a unit. The set  $U(\mathcal{A})$  of all invertible elements of  $\mathcal{A}$  forms a Moufang loop with respect to the multiplication [32]. Conversely, not any Moufang loop can be imbedded into a loop of type  $U(\mathcal{A})$  for a suitable unital alternative algebra  $\mathcal{A}$ . A counter-example was given in [32]. In [31], the author characterizes the Moufang loops which are imbeddable into a loop of type  $U(\mathcal{A})$ .

The Moufang algebras which are the corresponding algebras of a Moufang loop are defined as follows:

**Definition 3.6** A left Moufang algebra  $(\mathcal{A}, \mu)$  is one which is left alternative and satisfying the Moufang identity that is

$$\mu(\mu(x, y), \mu(z, x)) = \mu(\mu(x, \mu(y, z)), x). \quad (12)$$

The Moufang identities (8, 9, 10) are expressed in terms of associator by

$$\mathfrak{as}(x, y, z \cdot x) = x \cdot \mathfrak{as}(y, z, x) \quad (13)$$

$$\mathfrak{as}(x \cdot y, z, x) = \mathfrak{as}(x, y, z) \cdot x \quad (14)$$

$$\mathfrak{as}(y, x^2, z) = x \cdot \mathfrak{as}(y, x, z) + \mathfrak{as}(y, x, z) \cdot x \quad (15)$$

It turns out that in characteristic different from 2, all left alternative algebras are left Moufang algebras. Also, a left Moufang algebra is alternative if and only if it is flexible, that is  $\mathfrak{as}(x, y, x) = 0$  for all  $x, y \in \mathcal{A}$ . In [33] it shown that Malcev algebras form a class of a so-called General Lie triple system, called also Lie-Yamaguti algebras (and contains Lie triple system).

**Definition 3.7** [33] A Lie-Yamaguti algebra or General Lie triple system is an algebra  $\mathcal{A}$  over a field  $\mathbb{K}$  with a  $\mathbb{K}$ -trilinear map denoted  $\tau(-, -, -)$  satisfying the following conditions:

$$\mu(x, x) = 0, \quad (16)$$

$$\tau(x, x, y) = 0, \quad (17)$$

$$\tau(x, y, z) + \tau(y, z, x) + \tau(z, x, y) + \mu(\mu(x, y), z) + \mu(\mu(y, z), x) + \mu(\mu(z, x), y) = 0. \quad (18)$$

$$\tau(\mu(x, y), z, w) + \tau(\mu(y, z), x, w) + \tau(\mu(z, x), y, w) = 0 \quad (19)$$

$$\tau(x, y, \mu(z, w)) = \mu(\tau(x, y, z), w) + \mu(z, \tau(x, y, w)) \quad (20)$$

$$\tau(x, y, \tau(z, v, w)) = \tau(\tau(x, y, z), v, w) + \tau(z, \tau(x, y, v), w) + \tau(z, v, \tau(x, y, w)). \quad (21)$$

Any Lie algebra with Jacobi bracket  $[-, -]$  can be Lie-Yamaguti algebra by putting  $\mu(x, y) := [x, y]$  and  $\tau(x, y, z) := [[x, y], z]$ . Clearly, if  $\tau(x, y, z) = 0$  for all  $x, y, z \in \mathcal{A}$ , then the Lie-Yamaguti algebra reduces to a Lie algebra. And if  $\mu(x, y) = 0$  for all  $x, y \in \mathcal{A}$ , then the Lie-Yamaguti algebra reduces to a Lie triple system (see [20] for the definition of Lie triple system).

The alternative algebras are connected to Jordan algebras as follows. Given an alternative algebra  $(\mathcal{A}, \mu)$  then  $(\mathcal{A}, \mu^+)$ , where  $\mu^+(x, y) = \mu(x, y) + \mu(y, x)$ , is a Jordan algebra, that is the commutative multiplication  $\mu^+$  satisfies the identity  $\mathfrak{a}_{\mu^+}(x^2, y, x) = 0$ .

## 4 Cohomology of left alternative algebras

Let  $\mathcal{A}$  be a left alternative  $\mathbb{K}$ -algebra defined by a multiplication  $\mu$ . A left alternative  $p$ -cochain is a linear map from  $\mathcal{A}^{\otimes p}$  to  $\mathcal{A}$ . We denote by  $\mathcal{C}^p(\mathcal{A}, \mathcal{A})$  the group of all  $p$ -cochains.

### 4.1 First differential map

Let  $\text{id}$  denotes the identity map on  $\mathcal{A}$ . For  $f \in \mathcal{C}^1(\mathcal{A}, \mathcal{A})$ , we define the first differential  $\delta^1 f \in \mathcal{C}^2(\mathcal{A}, \mathcal{A})$  by

$$\delta^1 f = \mu \circ (f \otimes \text{id}) + \mu \circ (\text{id} \otimes f) - f \circ \mu. \quad (22)$$

We remark that the first differential of a left alternative algebra is similar to the first differential map of Hochschild cohomology of an associative algebra (1-cocycles are derivations).

### 4.2 Second differential map

Let  $\phi \in \mathcal{C}^2(\mathcal{A}, \mathcal{A})$ , we define the second differential  $\delta^2 \phi \in \mathcal{C}^3(\mathcal{A}, \mathcal{A})$  by,

$$\delta^2 \phi = [\mu \circ (\phi \otimes \text{id} - \text{id} \otimes \phi) + \phi \circ (\mu \otimes \text{id} - \text{id} \otimes \mu)] \circ (\text{id}^{\otimes 3} + \sigma_1). \quad (23)$$

where  $\sigma_1$  is defined on  $\mathcal{A}^{\otimes 3}$  by  $\sigma_1(x \otimes y \otimes z) = y \otimes x \otimes z$ .

**Remark 4.1** The left alternative algebra 2-differential defined in (23) may be written using the Hochschild differential  $\delta_H^2$  as

$$\delta^2 \phi = \delta_H^2 \phi \circ (\text{id}^{\otimes 3} + \sigma_1) \quad (24)$$

**Proposition 4.2** *The composite  $\delta^2 \circ \delta^1$  is zero.*

*Proof.* Let  $x, y, z \in \mathcal{A}$  and  $f \in \mathcal{C}^1(\mathcal{A}, \mathcal{A})$ ,

$$\delta^1 f(x \otimes y) = \mu(f(x) \otimes y) + \mu(x \otimes f(y)) - f(\mu(x \otimes y)).$$

Then

$$\begin{aligned}
\delta^2(\delta^1 f)(x \otimes y \otimes z) &= (xy)f(z) - f((xy)z) + [f(xy)]z + [xf(y)]z - [f(xy)]z + [(f(x))y]z + \\
&\quad + (yx)f(z) - f((yx)z) + [f(yx)]z + [yf(x)]z - [f(yx)]z + [(f(y))x]z + \\
&\quad - \{xf(yz) - f(x[yz]) + [f(x)](yz) + x(yf(z)) - xf(yz) + x[(f(y))z] + \\
&\quad + yf(xz) - f(y[xz]) + [f(y)](xz) + y(xf(z)) - yf(xz) + y[(f(x))z]\} \\
&= [(xy)f(z) + (yx)f(z) - x(yf(z)) - y(xf(z))] - [f((xy)z) + f((yx)z) + \\
&\quad - f(x(yz)) - f(y(xz))] + [(xf(y))z + (f(y)x)z - (f(y))(xz) - x(f(y)z)] + \\
&\quad + [(f(x)y)z + (yf(x))z - (f(x))(yz) - y(f(x)z)] \\
&= 0.
\end{aligned}$$

After simplifying the terms which cancel in pairs, we group the remaining ones into brackets so each bracket cancels using the left alternative algebra axiom (equation (3)).  $\square$

**Example 4.3** Let  $\mathcal{A} = \mathcal{M}_2(\mathbb{K})$  denotes the associative algebra of 2 by 2 matrices over the field  $\mathbb{K}$ , considered as left alternative algebra of dimension 4. Let  $e_1, e_2, e_3$  and  $e_4$  be a basis of  $\mathcal{A}$ . The second cohomology  $H^2(\mathcal{A}, \mathcal{A})$  is three-dimensional generated by  $[f_1], [f_2]$  and  $[f_3]$  where

$$\begin{aligned}
f_1(e_2 \otimes e_4) &= e_1, \quad f_1(e_3 \otimes e_2) = -e_3, \quad f_1(e_4 \otimes e_1) = e_3, \quad f_1(e_4 \otimes e_2) = e_4, \\
f_2(e_2 \otimes e_3) &= e_2, \quad f_2(e_3 \otimes e_1) = -e_4, \quad f_2(e_3 \otimes e_3) = e_3, \quad f_2(e_3 \otimes e_4) = e_4, \\
f_3(e_2 \otimes e_3) &= e_1, \quad f_3(e_3 \otimes e_2) = e_4.
\end{aligned}$$

The non-specified terms of these generators are zeros. These generators were obtained independently using the softwares Maple and Mathematica.

### 4.3 Third differential map

Let  $\psi \in \mathcal{C}^3(\mathcal{A}, \mathcal{A})$ , we define the third differential  $\delta^3\psi \in \mathcal{C}^4(\mathcal{A}, \mathcal{A})$  as,

$$\begin{aligned}
\delta^3\psi &= \mu(\psi \otimes \text{id})(\text{id}^{\otimes 3} - \sigma_1) + \mu(\text{id} \otimes \psi)(\text{id}^{\otimes 3} - \sigma_2) \\
&\quad - \psi(\mu \otimes \text{id}^{\otimes 2})(\text{id}^{\otimes 3} + \sigma_2 \circ \sigma_1) + \psi(\text{id} \otimes \mu \otimes \text{id})(\text{id}^{\otimes 3} + \sigma_1 \circ \sigma_2) - \psi(\text{id}^{\otimes 2} \otimes \mu)(\text{id}^{\otimes 3} - \sigma_1).
\end{aligned}$$

That is for all  $\psi \in \mathcal{C}^3(\mathcal{A}, \mathcal{A})$  and  $x_1, \dots, x_4 \in \mathcal{A}$

$$\begin{aligned}
\delta^3\psi(x_1, x_2, x_3, x_4) &= \mu(x_1 \otimes \psi(x_2 \otimes x_3 \otimes x_4)) - \mu(x_1 \otimes \psi(x_3 \otimes x_2 \otimes x_4)) \\
&\quad + \mu(\psi(x_1 \otimes x_2 \otimes x_3) \otimes x_4) - \mu(\psi(x_2 \otimes x_1 \otimes x_3) \otimes x_4) \\
&\quad - \psi(\mu(x_1 \otimes x_2) \otimes x_3 \otimes x_4) - \psi(\mu(x_2 \otimes x_3) \otimes x_1 \otimes x_4) \\
&\quad + \psi(x_1 \otimes \mu(x_2 \otimes x_3) \otimes x_4) + \psi(x_3 \otimes \mu(x_1 \otimes x_2) \otimes x_4) \\
&\quad - \psi(x_1 \otimes x_2 \otimes \mu(x_3 \otimes x_4)) + \psi(x_2 \otimes x_1 \otimes \mu(x_3 \otimes x_4)).
\end{aligned}$$

**Proposition 4.4** *The composite  $\delta^3 \circ \delta^2$  is zero.*

*Proof.* Let  $x_1, \dots, x_4 \in \mathcal{A}$  and  $f \in \mathcal{C}^2(\mathcal{A}, \mathcal{A})$ . Then, by substituting  $\psi$  with  $\delta^2 f$  in the previous formula and rearranging the terms we get

$$\begin{aligned} \delta^3(\delta^2 f)(x_1 \otimes x_2 \otimes x_3 \otimes x_4) &= x_1[\delta^2 f(x_2 \otimes x_3 \otimes x_4) - \delta^2 f(x_3 \otimes x_2 \otimes x_4)] \\ &\quad - [\delta^2 f(x_1 x_2 \otimes x_3 \otimes x_4) - \delta^2 f(x_3 \otimes x_1 x_2 \otimes x_4)] \\ &\quad + [\delta^2 f(x_1 \otimes x_2 x_3 \otimes x_4) - \delta^2 f(x_2 x_3 \otimes x_1 \otimes x_4)] \\ &\quad - [\delta^2 f(x_1 \otimes x_2 \otimes x_3 x_4) - \delta^2 f(x_2 \otimes x_1 \otimes x_3 x_4)] \\ &\quad + [\delta^2 f(x_1 \otimes x_2 \otimes x_3) - \delta^2 f(x_2 \otimes x_1 \otimes x_3)] x_4 \\ &= 0, \end{aligned}$$

since  $\delta^2 f(x \otimes y \otimes z) = \delta^2 f(y \otimes x \otimes z)$ , for all  $x, y, z \in \mathcal{A}$ .  $\square$

It is an interesting problem to find the higher  $p^{th}$  differential maps and study the properties of the cohomology groups. This will be considered by the authors in a forthcoming work.

## 5 Formal Deformations of left alternative algebras

Let  $(\mathcal{A}, \mu_0)$  be a left alternative algebra. Let  $\mathbb{K}[[t]]$  be the power series ring in one variable  $t$  and coefficients in  $\mathbb{K}$  and  $\mathcal{A}[[t]]$  be the set of formal power series whose coefficients are elements of  $\mathcal{A}$  (note that  $\mathcal{A}[[t]]$  is obtained by extending the coefficients domain of  $\mathcal{A}$  from  $\mathbb{K}$  to  $\mathbb{K}[[t]]$ ). Then  $\mathcal{A}[[t]]$  is a  $\mathbb{K}[[t]]$ -module. When  $\mathcal{A}$  is finite-dimensional, we have  $\mathcal{A}[[t]] = \mathcal{A} \otimes_{\mathbb{K}} \mathbb{K}[[t]]$ . One notes that  $V$  is a submodule of  $\mathcal{A}[[t]]$ . Given a  $\mathbb{K}$ -bilinear map  $f : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ , it admits naturally an extension to a  $\mathbb{K}[[t]]$ -bilinear map  $f : \mathcal{A}[[t]] \otimes \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]]$ , that is, if  $x = \sum_{i \geq 0} a_i t^i$  and  $y = \sum_{j \geq 0} b_j t^j$  then  $f(x \otimes y) = \sum_{i \geq 0, j \geq 0} t^{i+j} f(a_i \otimes b_j)$ .

**Definition 5.1** Let  $(\mathcal{A}, \mu_0)$  be a left alternative algebra. A *formal left alternative deformation* of  $\mathcal{A}$  is given by the  $\mathbb{K}[[t]]$ -bilinear map  $\mu_t : \mathcal{A}[[t]] \otimes \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]]$  of the form  $\mu_t = \sum_{i \geq 0} \mu_i t^i$ , where each  $\mu_i$  is a  $\mathbb{K}$ -bilinear map  $\mu_i : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$  (extended to be  $\mathbb{K}[[t]]$ -bilinear), such that for  $x, y, z \in \mathcal{A}$ , the following formal left alternativity condition holds

$$\mu_t(x \otimes \mu_t(y \otimes z)) - \mu_t(\mu_t(x \otimes y) \otimes z) + \mu_t(y \otimes \mu_t(x \otimes z)) - \mu_t(\mu_t(y \otimes x) \otimes z) = 0. \quad (25)$$

### 5.1 Deformation equation and Obstructions

The first problem is to give conditions about  $\mu_i$  such that the deformation  $\mu_t$  be alternative. Expanding the left side of the equation (25) and collecting the coefficients of  $t^k$  yields

$$\begin{cases} \sum_{i+j=k, i,j \geq 0} \mu_i(x \otimes \mu_j(y \otimes z)) - \mu_i(\mu_j(x \otimes y) \otimes z) + \mu_i(y \otimes \mu_j(x \otimes z)) - \mu_i(\mu_j(y \otimes x) \otimes z) = 0, \\ k = 0, 1, 2, \dots \end{cases}$$

This infinite system, called the *deformation equation*, gives the necessary and sufficient conditions for the left alternativity of  $\mu_t$ . It may be written

$$\begin{cases} \sum_{i=0}^k \mu_i(x \otimes \mu_{k-i}(y \otimes z)) - \mu_i(\mu_{k-i}(x, y) \otimes z) + \mu_i(y \otimes \mu_{k-i}(x \otimes z)) - \mu_i(\mu_{k-i}(y \otimes x) \otimes z) = 0, \\ k = 0, 1, 2, \dots \end{cases} \quad (26)$$



The first equation ( $k = 0$ ) is the left alternativity condition for  $\mu_0$ .

The second shows that  $\mu_1$  must be a 2-cocycle for the Alternative algebra cohomology defined above ( $\mu_1 \in Z^2(\mathcal{A}, \mathcal{A})$ ).

More generally, suppose that  $\mu_p$  be the first non-zero coefficient after  $\mu_0$  in the deformation  $\mu_t$ . This  $\mu_p$  is called the *infinitesimal* of  $\mu_t$ .

**Theorem 5.2** *The map  $\mu_p$  is a 2-cocycle of the left alternative algebras cohomology of  $\mathcal{A}$  with coefficient in itself.*

*Proof.* In the equation (26) make the following substitution  $k = p$  and  $\mu_1 = \dots = \mu_{p-1} = 0$ .  $\square$

**Definition 5.3** The 2-cocycle  $\mu_p$  is said integrable if it is the first non-zero term, after  $\mu_0$ , of a left alternative deformation.

The integrability of  $\mu_p$  implies an infinite sequence of relations which may be interpreted as the vanishing of the obstruction to the integration of  $\mu_p$ .

For an arbitrary  $k$ , with  $k > 1$ , the  $k^{th}$  equation of the system (26) may be written

$$\delta^2 \mu_k (x \otimes y \otimes z) = \sum_{i=1}^{k-1} \mu_i (\mu_{k-i} (x \otimes y) \otimes z) - \mu_i (x \otimes \mu_{k-i} (y \otimes z)) + \mu_i (\mu_{k-i} (y \otimes x) \otimes z) - \mu_i (y \otimes \mu_{k-i} (x \otimes z)).$$

Suppose that the truncated deformation  $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \dots + t^{m-1}\mu_{m-1}$  satisfies the deformation equation. The truncated deformation is extended to a deformation of order  $m$ , i.e.  $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \dots + t^{m-1}\mu_{m-1} + t^m\mu_m$  satisfying the deformation equation if

$$\delta^2 \mu_m (x \otimes y \otimes z) = \sum_{i=1}^{m-1} \mu_i (\mu_{m-i} (x \otimes y) \otimes z) - \mu_i (x \otimes \mu_{m-i} (y \otimes z)) + \mu_i (\mu_{m-i} (y \otimes x) \otimes z) - \mu_i (y \otimes \mu_{m-i} (x \otimes z)).$$

The right-hand side of this equation is called the *obstruction* to finding  $\mu_m$  extending the deformation.

We define a square operation on 2-cochains by

$$\mu_i \square \mu_j (x \otimes y \otimes z) = \mu_i (\mu_j (x \otimes y) \otimes z) - \mu_i (x \otimes \mu_j (y \otimes z)) + \mu_i (\mu_j (y \otimes x) \otimes z) - \mu_i (y \otimes \mu_j (x \otimes z)),$$

then the obstruction may be written  $\sum_{i=1}^{m-1} \mu_i \square \mu_{m-i}$  or  $\sum_{i+j=m} \mu_i \square \mu_j$ .

A straightforward computation gives the following

**Theorem 5.4** *The obstructions are left alternative 3-cocycles.*

**Remark 5.5** 1. The cohomology class of the element  $\sum_{i+j=m, i,j \neq m} \mu_i \square \mu_j$  is the first obstruction to the integrability of  $\mu_m$ .

Let us consider now how to extend an infinitesimal deformation to a deformation of order 2. Suppose

$m = 2$  and  $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2$ . The deformation equation of the truncated deformation of order 2 is equivalent to the finite system:

$$\begin{cases} \mu_0 \square \mu_0 &= 0 & (\mu_0 \text{ is left alternative}) \\ \delta\mu_1 &= 0 & (\mu_1 \in Z^2(\mathcal{A}, \mathcal{A})) \\ \mu_1 \square \mu_1 &= \delta\mu_2 \end{cases}$$

Then  $\mu_1 \square \mu_1$  is the first obstruction to integrate  $\mu_1$  and  $\mu_1 \square \mu_1 \in Z^3(\mathcal{A}, \mathcal{A})$ .

The elements  $\mu_1 \square \mu_1$  which are coboundaries permit to extend the deformation of order one to a deformation of order 2. But the elements of  $H^3(\mathcal{A}, \mathcal{A})$  gives the obstruction to the integrations of  $\mu_1$ .

2. If  $\mu_m$  is integrable then the cohomological class of  $\sum_{i+j=m, i,j \neq m} \mu_i \square \mu_j$  must be 0.

In the previous example  $\mu_1$  is integrable implies  $\mu_1 \square \mu_1 = \delta\mu_2$  which means that the cohomology class of  $\mu_1 \square \mu_1$  vanishes.

**Corollary 5.6** *If  $H^3(\mathcal{A}, \mathcal{A}) = 0$  then all obstructions vanish and every  $\mu_m \in Z^2(\mathcal{A}, \mathcal{A})$  is integrable.*

## 5.2 Equivalent and trivial deformations

In this section, we characterize equivalent as well as trivial deformations of left alternative algebras.

**Definition 5.7** Let  $(\mathcal{A}, \mu_0)$  be a left alternative algebra and let  $(\mathcal{A}_t, \mu_t)$  and  $(\mathcal{A}'_t, \mu'_t)$  be two left alternative deformations of  $\mathcal{A}$ , where  $\mu_t = \sum_{i \geq 0} t^i \mu_i$  and  $\mu'_t = \sum_{i \geq 0} t^i \mu'_i$ , with  $\mu_0 = \mu'_0$ .

We say that the two deformations are *equivalent* if there exists a formal isomorphism  $\Phi_t : \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]]$ , i.e. a  $\mathbb{K}[[t]]$ -linear map that may be written in the form  $\Phi_t = \sum_{i \geq 0} t^i \Phi_i = \text{id} + t\Phi_1 + t^2\Phi_2 + \dots$ , where  $\Phi_i \in \text{End}_{\mathbb{K}}(\mathcal{A})$  and  $\Phi_0 = \text{id}$  are such that the following relations hold

$$\Phi_t \circ \mu_t = \mu'_t \circ (\Phi_t \otimes \Phi_t). \quad (27)$$

A deformation  $\mathcal{A}_t$  of  $\mathcal{A}_0$  is said to be *trivial* if and only if  $\mathcal{A}_t$  is equivalent to  $\mathcal{A}_0$  (viewed as a left alternative algebra on  $\mathcal{A}[[t]]$ ).

We discuss in the following the equivalence of two deformations. Condition (27) may be written as

$$\Phi_t(\mu_t(x \otimes y)) = \mu'_t(\Phi_t(x) \otimes \Phi_t(y)), \quad \forall x, y \in \mathcal{A}. \quad (28)$$

Equation (28) is equivalent to

$$\sum_{i \geq 0} \Phi_i \left( \sum_{j \geq 0} \mu_j(x \otimes y) t^j \right) t^i = \sum_{i \geq 0} \mu'_i \left( \sum_{j \geq 0} \Phi_j(x) t^j \otimes \sum_{k \geq 0} \Phi_k(y) t^k \right) t^i \quad (29)$$

or

$$\sum_{i,j \geq 0} \Phi_i(\mu_j(x \otimes y)) t^{i+j} = \sum_{i,j,k \geq 0} \mu'_i(\Phi_j(x) \otimes \Phi_k(y)) t^{i+j+k}.$$

By identification of the coefficients, one obtains that the constant coefficients are identical, i.e.

$$\mu_0 = \mu'_0 \quad \text{because} \quad \Phi_0 = \text{id}.$$

For the coefficients of  $t$  one finds

$$\Phi_0(\mu_1(x \otimes y)) + \Phi_1(\mu_0(x \otimes y)) = \mu'_1(\Phi_0(x) \otimes \Phi_0(y)) + \mu'_0(\Phi_1(x) \otimes \Phi_0(y)) + \mu'_0(\Phi_0(x) \otimes \Phi_1(y)). \quad (30)$$

Since  $\Phi_0 = id$ , it follows that

$$\mu_1(x, y) + \Phi_1(\mu_0(x \otimes y)) = \mu'_1(x \otimes y) + \mu_0(\Phi_1(x) \otimes y) + \mu_0(x \otimes \Phi_1(y)). \quad (31)$$

Consequently,

$$\mu'_1(x \otimes y) = \mu_1(x \otimes y) + \Phi_1(\mu_0(x \otimes y)) - \mu_0(\Phi_1(x) \otimes y) - \mu_0(x \otimes \Phi_1(y)). \quad (32)$$

The second order conditions of the equivalence between two deformations of a left alternative algebra are given by (32) which may be written

$$\mu'_1(x \otimes y) = \mu_1(x \otimes y) - \delta^1 \Phi_1(x \otimes y). \quad (33)$$

In general, if the deformations  $\mu_t$  and  $\mu'_t$  of  $\mu_0$  are equivalent then  $\mu'_1 = \mu_1 + \delta^1 f_1$ .

Therefore, we have the following proposition:

**Proposition 5.8** *The integrability of  $\mu_1$  depends only on its cohomology class.*

Recall that two elements are cohomologous if their difference is a coboundary.

The equation  $\delta^2 \mu_1 = 0$  implies that  $\delta^2 \mu'_1 = \delta^2 (\mu_1 + \delta^1 f_1) = \delta^1 \mu_1 + \delta^2 (\delta^1 f_1) = 0$ .

If  $\mu_1 = \delta^1 g$  then  $\mu'_1 = \delta^1 g - \delta^1 f_1 = \delta^1 (g - f_1)$ .

Then if two integrable 2-cocycles are cohomologous, then the corresponding deformations are equivalent.

**Remark 5.9** Elements of  $H^2(\mathcal{A}, \mathcal{A})$  give the infinitesimal deformations ( $\mu_t = \mu_0 + t\mu_1$ ).

**Proposition 5.10** *Let  $(\mathcal{A}, \mu_0)$  be a left alternative algebra. There is, over  $\mathbb{K}[[t]]/t^2$ , a one-to-one correspondence between the elements of  $H^2(\mathcal{A}, \mathcal{A})$  and the infinitesimal deformation of  $\mathcal{A}$  defined by*

$$\mu_t(x \otimes y) = \mu_0(x \otimes y) + t\mu_1(x \otimes y), \quad \forall x, y \in \mathcal{A}. \quad (34)$$

*Proof.* The deformation equation is equivalent to  $\delta^2 \mu_1 = 0$ , that is  $\mu_1 \in Z^2(\mathcal{A}, \mathcal{A})$ .  $\square$

**Theorem 5.11** *Let  $(\mathcal{A}, \mu_0)$  be a left alternative algebra and  $\mu_t$  be a one parameter family of deformation of  $\mu_0$ . Then  $\mu_t$  is equivalent to  $\mu_t = \mu_0 + t^p \mu'_p + t^{p+1} \mu'_{p+1} + \dots$ , where  $\mu'_p \in Z^2(\mathcal{A}, \mathcal{A})$  and  $\mu'_p \notin B^2(\mathcal{A}, \mathcal{A})$ .*

*Proof.* Suppose now that  $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \dots$ , is a one parameter family of deformation of  $\mu_0$  for which  $\mu_1 = \dots = \mu_{m-1} = 0$ . The deformation equation implies  $\delta\mu_m = 0$  ( $\mu_m \in Z^2(\mathcal{A}, \mathcal{A})$ ). If further  $\mu_m \in B^2(\mathcal{A}, \mathcal{A})$  (ie.  $\mu_m = \delta g$ ), then setting the morphism  $f_t = id + tf_m$ , we have, for all  $x, y \in \mathcal{A}$ ,

$$\mu'_t(x \otimes y) = f_t^{-1} \circ \mu_t \circ (f_t(x) \otimes f_t(y)) = \mu_0(x \otimes y) + t^{m+1} \mu_{m+1}(x \otimes y) \dots$$

And again  $\mu_{m+1} \in Z^2(\mathcal{A}, \mathcal{A})$ .  $\square$

**Corollary 5.12** *If  $H^2(\mathcal{A}, \mathcal{A}) = 0$ , then all deformations of  $\mathcal{A}$  are equivalent to a trivial deformation.*

In fact, assume that there exists a non trivial deformation of  $\mu_0$ . Following the previous theorem, this deformation is equivalent to  $\mu_t = \mu_0 + t^p \mu'_p + t^{p+1} \mu'_{p+1} + \dots$  where  $\mu'_p \in Z^2(\mathcal{A}, \mathcal{A})$  and  $\mu'_p \notin B^2(\mathcal{A}, \mathcal{A})$ . But this is impossible because  $H^2(\mathcal{A}, \mathcal{A}) = 0$ .

**Remark 5.13** A left alternative algebra for which every formal deformation is equivalent to a trivial deformation is called rigid. The previous corollary provide a sufficient condition for a left alternative algebra to be rigid. In general this condition is not necessary.

**Acknowledgment** M. E. would like to thank S. Carter and M. Saito for fruitfull discussions.

## References

- [1] Albuquerque H., and Majid S., *Quasialgebra Structure of the Octonions*, J. Algebra **220** (1999), 188–224.
- [2] Ataguema H., and Makhlouf A., *Deformations of ternary algebras*, J. Gen. Lie Theory Appl. 1 (2007), no. 1, 41–55.
- [3] Ataguema H., and Makhlouf A., *Notes on cohomology of ternary algebras of associative type*, J. Gen. Lie Theory Appl. **3**, no 3 (2009), 1–19.
- [4] Baez J.C., *The octonions*, Bull. of the Amer. Math. Soc., **39** 2, (2001), 145–2005.
- [5] Bordemann M., Makhlouf A., and Petit T.: *Déformation par Quantification et Rigidité des Algèbres Enveloppantes*, J. of Algebra, **285**, (2005), 623–648.
- [6] Carter J.S., Crans A., Elhamdadi M., and Saito M., *Cohomology of the adjoint of Hopf Algebras*, J. Gen. Lie Theory Appl. **2**, no 1 (2008), 19–34.
- [7] Carter J.S., Crans A., Elhamdadi M., and Saito M., *Cohomology of categorical self-distributivity*, J. Homotopy Relat. Struct. **3** (2008), no. 1, 13–63.
- [8] Dzhumadil'daev A., and Zusmanovich P. : *The alternative operad is not Koszul*, arXiv:0906.1272 [Math.RA], (2009).
- [9] Fialowski A., *Deformation of Lie algebras*, Math USSR Sbornik, vol. **55**, (1986), 467–473.
- [10] Fialowski A., *An example of Formal Deformations of Lie Algebras*, In: Deformation theory of algebras and structures and applications, ed. Hazewinkel and Gerstenhaber, NATO Adv. Sci. Inst. Serie C, 297, Kluwer Acad. Publ., (1988), 375–401.
- [11] Fialowski A., and O'Halloran J.: *A Comparison of Deformations and Orbit Closure*, Comm. Algebra **18**, (1990), 4121–4140.
- [12] Gerstenhaber M., *On the deformation of rings and algebras*, Ann. of Math. (2) **79**, (1964), 59–103.
- [13] Gerstenhaber M., *On the deformation of rings and algebras. II*, Ann. of Math. (2) **84** (1966) 1–19.
- [14] Gerstenhaber M., *On the deformation of rings and algebras III*, Ann. of Math. (2) **88** (1968) 1–34.
- [15] Gerstenhaber M., *On the deformation of rings and algebras IV*, Ann. of Math. (2) **99** (1974) 257–276.
- [16] Goodaire E., *Alternative rings of small order and the hunt for Moufang circle loops*, Nonassociative algebra and its applications (São Paulo, 1998), 137–146, Lecture Notes in Pure and Appl. Math., 211, Dekker, New York, 2000.
- [17] Goze M., and Makhlouf A., *On the complex rigid associative algebras*, Comm. Algebra **18** (1990), no 12, 4031–4046.

- [18] Goze M., *Perturbations of Lie algebra structures*, In Deformation theory of algebras and structures and applications, ed. Hazewinkel and Gerstenhaber, NATO Adv. Sci. Inst. Serie C 297, Kluwer Acad. Publ. (1988).
- [19] Goze M., and Remm E., *Valued deformations of algebras* J. of Algebra and its Appl. **3**, (2004), no 4, 345–365.
- [20] Jacobson N., *Lie and Jordan triples*, American Journal of Mathematics **71**, (1949) 149–170.
- [21] Kerdman, F. S., *Analytic Moufang loops in the large*, Algebra i Logica **18** (1979), 523–555.
- [22] Laudal O. A., *Formal moduli of algebraic structures*, Lecture Notes in Mathematics, **754**, Springer, Berlin, 1979.
- [23] Makhlouf A., *A comparison of deformations and geometric study of varieties of associative algebras*, International Journal of Mathematics and Mathematical Science Vol. 2007 , Article ID 18915,(2007).
- [24] Makhlouf A., *Degeneration, rigidity and irreducible components of Hopf algebras*, Algebra Colloquium, **12**,(2005) no 2, 241–254.
- [25] Makhlouf A., and Goze M., *Classification of Rigid Associative Algebras in low Dimensions*, in *Lois d'algèbres et variétés algébriques*, Collection travaux en cours, 50, Hermann Paris, (1996) 5–22.
- [26] Maltsev, A. I., *Analytical loops*, Matem. Sbornik. **36** (1955), 569-576 (in Russian).
- [27] Markl M., Stasheff J.D., *Deformation theory via deviations*, J. Algebra **170**, (1994), 122–155.
- [28] McCrimmon K., *Alternative algebras*, <http://www.mathstat.uottawa.ca/~neher/Papers/alternative/>
- [29] Paal E., *Note on analytic Moufang loops*, Comment. Math. Univ. Carolinae, **45** no 2, (2004), 349–354.
- [30] Paal E., *Moufang loops and generalized Lie-Cartan theorem*, Journal of Gen. Lie Theory and Applications, **2** (2008), 45–49.
- [31] Sandu N. I., *About the embedding of Moufang loops in alternative algebras II*, arXiv:0804.2049v1 [math.RA], (2008).
- [32] Shestakov I.P., *Moufang Loops and alternative algebras*, Proc. of Amer. Math. Soc., **132**, no 2, (2003), 313–316.
- [33] Yamaguti K., *On the theory of Malcev algebras* Kumamoto J. Sci. Ser. A **6** (1963) 9–45.